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Some vector-valued theta series on $U(2,2)$ and $Sp(1,1)$

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This is an overview of the paper [5], which is a joint work with Hiro-aki NARITA.

In this talk, we will first construct vector-valued (holomorphic) singular modular forms on $U(2,2)$, which can be regarded as generalizations of theta series. The restrictions of these singular forms to $Sp(1,1)$ are modular forms which generate quaternionic discrete series.

1 The construction of vector-valued singular forms on $U(2,2)$

First of all, let us construct vector-valued singular modular forms on $U(2,2)$. Put

$$U(2,2)=\left\{g\in GL(4,\mathbb{C})\,\middle|\,g\begin{pmatrix}0&-1_2\\1_2&0\end{pmatrix}{}^t\overline{g}=\begin{pmatrix}0&-1_2\\1_2&0\end{pmatrix}\right\},$$

$$\mathfrak{H}_{U(2,2)}=\left\{z\in M_2(\mathbb{C})\,\middle|\,\sqrt{-1}({}^t\overline{z}-z)>0\right\}.$$

As is well-known, the group $U(2,2)$ acts on $\mathfrak{H}_{U(2,2)}$ as $\begin{pmatrix}\alpha_1&\alpha_2\\\alpha_3&\alpha_4\end{pmatrix}(z)=(\alpha_1z+\alpha_2)(\alpha_3z+\alpha_4)^{-1}$, and we put $\mu_1\left(\begin{pmatrix}\alpha_1&\alpha_2\\\alpha_3&\alpha_4\end{pmatrix},z\right)=\alpha_3z+\alpha_4$.

For any non-negative integer κ, we denote by V_κ the space of homogeneous polynomials of degree κ with two variables. (Note that $\dim_{\mathbb{C}}V_\kappa=\kappa+1$.) We define a map $\sigma_\kappa:M_2(\mathbb{C})\to\mathrm{End}(V_\kappa)$ by

$$(\sigma_\kappa(h)v)(X,Y)=v(aX+cY,bX+dY)\qquad\left(h=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_2(\mathbb{C})\right).$$

This induces an irreducible rational representation (σ_κ,V_κ) of $GL_2(\mathbb{C})$. The space V_κ has a basis $\{v_i=v_i(X,Y):=X^iY^{\kappa-i}\,|\,0\le i\le\kappa\}$. The vector v_κ is a highest weight vector.

Take an imaginary quadratic field K and fix it. For any $r,s\in K^2$ and any $\mathbb{Z}$-lattice L in K^2, we define a V_κ-valued holomorphic function $\theta^{(\kappa)}(z;L,r,s)$ on $z\in\mathfrak{H}_{U(2,2)}$ as

$$\theta^{(\kappa)}(z;L,r,s)=\sum_{x\in L+r}\exp\left(\pi\sqrt{-1}\mathrm{Tr}_{k/\mathbb{Q}}({}^t\overline{x}s)\right)\exp\left(\pi\sqrt{-1}\,{}^t\overline{x}zx\right)\sigma_\kappa\left((x,*)\right)\cdot v_\kappa.$$

Here $(x, *)$ denotes an arbitrary 2×2-matrix whose first column is x, since $\sigma_\kappa((x, *)) \cdot v_\kappa$ does not depend on $*$.

Theorem 1.1. *The function $\theta^{(\kappa)}(z; L, r, s)$ is a modular form of weight $\sigma_\kappa \otimes \det$. It means, there exists a congruence subgroup Γ of $U(2,2:K)(:= U(2,2) \cap GL_4(K))$ so that*

$$\theta^{(\kappa)}(\gamma(z); L, r, s) = \det(\mu_1(\gamma, z))\sigma_\kappa(\mu_1(\gamma, z))\theta^{(\kappa)}(z; L, r, s)$$

for any $\gamma \in \Gamma$.

Remark 1. Note that $\theta^{(\kappa)}(z; L, r, s)$ is so-called a singular form since all of its Fourier coefficients at non-degenerate indices are 0.
Remark 2. In case $\kappa = 0$, the scalar valued function $\theta^{(0)}(z; L, r, s)$ is a pull-back of a theta series on $Sp(4, \mathbb{Q})$, and a modular form of weight 1.

This theorem can be proved in the same way as the proof of automorphy of scalar-valued theta series.

2 Restriction to $Sp(1,1)$

Let B be a definite quaternion algebra over $\mathbb{Q}$, and $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$ be the Hamilton quaternion algebra. Since $B \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}$, we regard B as a dense subset of $\mathbb{H}$. The main involution of $\mathbb{H}$ is written as $\tau \to \overline{\tau}$ for $\tau \in \mathbb{H}$, and the reduced norm and the reduced trace are defined as $N(\tau) = \tau\overline{\tau}$ and $\mathrm{tr}(\tau) = \tau + \overline{\tau}$. We write $\mathbb{H}^- = \{\tau \in \mathbb{H} \,|\, \mathrm{tr}(\tau) = 0\}$ and $B^- = B \cap \mathbb{H}^-$.

Put

$$Sp(1,1)_{\mathbb{R}} := \left\{ g \in M_2(\mathbb{H}) \;\middle|\; g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\overline{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$
$$Sp(1,1)_{\mathbb{Q}} = Sp(1,1)_{\mathbb{R}} \cap M_2(B).$$

The canonical maximal compact subgroup of $Sp(1,1)_{\mathbb{R}}$ is given as

$$K_{Sp(1,1)} = \left\{ g \in Sp(1,1)_{\mathbb{R}} \;\middle|\; g = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \text{ with } a, b \in \mathbb{H} \right\}.$$

Hence the symmetric space of $Sp(1,1)_{\mathbb{R}}$ is

$$\mathfrak{H}_{Sp(1,1)} = \{\tau \in \mathbb{H} \,|\, \mathrm{tr}(\tau) > 0\} \cong Sp(1,1)_{\mathbb{R}}/K_{Sp(1,1)},$$

and the factor of automorphy $\mu_2(g, \tau)$ is given as

$$\mu_2\left(\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \tau \right) = \alpha_3\tau + \alpha_4 \qquad \text{for } \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in Sp(1,1)_{\mathbb{R}} \text{ and } \tau \in \mathfrak{H}_{Sp(1,1)}.$$

For given integer $\kappa \geq 2$, $\xi \in B^-$, $p, q \in B$ and $\mathbb{Z}$-lattice $\Lambda \subset B$, we define a V_κ-valued function $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ on $g \in Sp(1,1)_{\mathbb{R}}$ as

$$\begin{aligned}
&\theta_\xi^{(\kappa)}\left(\begin{pmatrix} \sqrt{y} & x(\sqrt{y})^{-1} \\ 0 & (\sqrt{y})^{-1} \end{pmatrix} k_\infty; \Lambda, p, q\right) \\
&:= \sum_{\lambda \in \Lambda + p} \exp\left(\pi\sqrt{-1}\mathrm{tr}(\bar{\lambda} q)\right) y^{\frac{\kappa}{2}+1} \exp\left(-4\pi\sqrt{N(\lambda\xi\bar{\lambda})}y\right) \\
&\quad \exp(2\pi\sqrt{-1}\mathrm{tr}(\lambda\xi\bar{\lambda}x))\sigma_\kappa(\iota(\mu_2(k_\infty, 1)))^{-1}\sigma_\kappa(\iota(\lambda u_\xi))v_\kappa,
\end{aligned}$$

where $x \in \mathbb{H}^-$, $y \in \mathbb{R}_+$ and $k_\infty \in K_{Sp(1,1)}$. The symbol ι denotes the canonical embedding of $\mathbb{H}$ into $M_2(\mathbb{C})$, and $u_\xi \in \mathbb{H}$ is as $u_\xi i \overline{u_\xi} = \frac{\xi}{\sqrt{N(\xi)}}$.

Theorem 2.1. *The function $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ is a modular form of weight σ_κ with respect to some congruence subgroup of $Sp(1,1)_{\mathbb{Q}}$, generating a quaternionic discrete series.*

Sketch of proof. The automorphy of $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ comes from the fact that it is a pull-back of $\theta^{(\kappa)}(z; L, r, s)$ on $U(2,2:K)$ with $K := \mathbb{Q}(\sqrt{-N(\xi)}) \cong \mathbb{Q}(\xi)$ and suitable L, r, s.

Remark1. Note that the Fourier coefficients of $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ are 0 outside the single-orbit of indices $\{\lambda\xi\bar{\lambda} \,|\, \lambda \in B^\times\}$, (and so are those of $\theta_\xi^{(\kappa)}(\alpha g; \Lambda, p, q)$ for any $\alpha \in Sp(1,1)_{\mathbb{Q}}$.) We should add that the subalgebra $\mathbb{Q}(\lambda\xi\bar{\lambda})$ is isomorphic to $\mathbb{Q}(\xi)$ ($\cong K$) for any $\lambda \in B^\times$.

Remark2. We can take $p \in B$ so that $\theta_\xi^{(\kappa)}(g; \Lambda, p, q) \not\equiv 0$ for given ξ, Λ and q. In fact, there is $p \in B \setminus \{0\}$ which satisfies $N(p) < \frac{1}{4}N(l)$ for any $l \in \Lambda \setminus \{0\}$. Then the Fourier coefficient indexed by $p\xi\bar{p}$ is $\exp(\pi\sqrt{-1}\mathrm{tr}(\bar{p}q))\sigma_\kappa(\iota(pu_\xi))v_\kappa \neq 0$ since $N(p) < N(p+l)$ holds for any $l \in \Lambda \setminus \{0\}$.

In the end, we should note that the Fourier coefficients of $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ are essentially algebraic. Take any $0 \neq \eta \in B^-$ such that $\xi\eta \in B^-$. Then the set $\{1, \xi, \eta, \xi\eta\}$ is a basis of B over $\mathbb{Q}$. We take $u_{\xi,\eta} \in \mathbb{H}$ which satisfies $u_{\xi,\eta} i \overline{u_{\xi,\eta}} = \frac{\xi}{\sqrt{N(\xi)}}$ and $u_{\xi,\eta} j \overline{u_{\xi,\eta}} = \frac{\eta}{\sqrt{N(\eta)}}$. Note that such $u_{\xi,\eta}$ exists and is uniquely determined up to the multiple of $\{\pm 1\}$. From now on, we replace u_ξ by $u_{\xi,\eta}$ in the definition of $\theta_\xi^{(\kappa)}$. (By this replacement, the function $\theta_\xi^{(\kappa)}$ changes just a constant multiple.) We put $V_\kappa(\overline{\mathbb{Q}}) = \sum_{i=0}^{\kappa} \overline{\mathbb{Q}}v_i$ and $V_{\kappa,B}(\overline{\mathbb{Q}}) = \sigma_\kappa(\iota(u_{\xi,\eta}))V_\kappa(\overline{\mathbb{Q}})$. It can be verified that $V_{\kappa,B}(\overline{\mathbb{Q}})$ does not depend on the choice of (ξ, η). Then we have the following.

Corollary 2.2. *If we write the Fourier expansion of $\theta_\xi^{(\kappa)}(g; \Lambda, p, q)$ as*

$$\begin{aligned}
&\theta_\xi^{(\kappa)}\left(\begin{pmatrix} \sqrt{y} & x(\sqrt{y})^{-1} \\ 0 & (\sqrt{y})^{-1} \end{pmatrix} k_\infty; \Lambda, p, q\right) \\
= &\sum_{\delta \in B^-} \sigma_\kappa(\iota(\mu_2(k_\infty, 1)))^{-1} C_\delta y^{\frac{\kappa}{2}+1} \exp(-4\pi\sqrt{N(\delta)}y) \exp(2\pi\sqrt{-1}\mathrm{tr}(\delta x)),
\end{aligned}$$

with $C_\delta \in V_\kappa$, *then* $C_\delta \in V_{\kappa,B}(\overline{\mathbb{Q}})$ *for any* $\delta \in B^-$.

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